

SIMPLIFIED EQUATION OF MOTION OF A SUBMERGED OFFSHORE CABLE SUBJECTED TO TWO-DEGREE OF FREEDOM MASS SYSTEMS

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ABSTRACT

In this paper, a simplified equation of motion is developed to simulate a submerged cable subjected to two degree of freedom using method of influence coefficient. A submerged cable as a continuous system with distributed masses is idealized as a lumped mass system with distributed masses lumped at two chosen points called the nodal points. The lumped masses are assumed to undergo normal mode oscillation at the chosen nodal points. A numerical example is given to illustrate the performance and applicability of the present model. The natural frequencies of the submersed cable generated using the present model compared favourably with those of the exact solution (James 1984) showing the effectiveness of the present model in the analysis and design of submerged cables.

KEYWORDS: Dynamic model, simulate, submerged cable, Two-degree of freedom, natural frequencies, nodal points, submerged cables.

INTRODUCTION

Offshore cables are structural systems with distributed mass. They are used in many ocean engineering applications ranging from moorings, in which cable elements may be required to withstand large tensile loads in which cable elements may provide structural support. Cables are susceptible to dynamic response, the analysis of which is of paramount importance in design (Lee 2001; Nakajima et al.; 1982, Papzoglou et al., 1990; Watson and Kuneman 1975; Hoffman 1993; James 1984).

In this paper, an efficient and relatively simple method of dynamic analysis for the prediction of natural frequencies of a submerged cable is developed. The lumped mass method is used and the method of influence coefficient is employed to facilitate the needed solution. In this method, the continuous distribution of the cable mass is replaced by the discrete distribution of the lumped masses at the finite number of points on the cable, resulting in relatively simple effective analysis (Thomson 1988). Dynamic of a submersed cable system as a continuous system is dubious as it involves a lot of mathematical manipulations (Thomson 1988). The difficulty involved in the dynamic analysis of structural systems with distributed mass lies with the many degrees of freedom involved which implies many unknowns. The objective of the present study is to develop a dynamic model that simulates the dynamic response of a submersed cable under self-excited vibration. An example is presented to show the reliable performance of the present model and its applicability. The results obtained compared favourably with the simulated values. The model developed here is relatively simple and effective and can be used for the dynamic analysis of a submerged cable used for offshore operations.

MATERIALS AND METHODS

In the present study, the distributed mass of the cable is replaced by a discrete distribution of lumped masses at a finite number of points on the cable. The finite number of lumped masses are located at chosen nodal points. That replacement amounts to idealizing the submersed cable as a set of lumped masses at chosen points called the nodal points. The cable tension is assumed constant as the cable is under self-excited vibration.

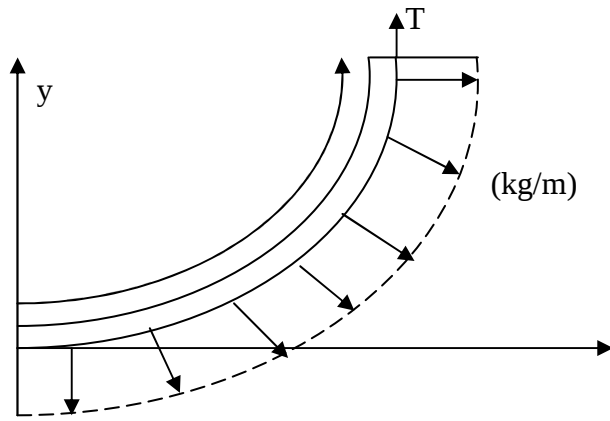


Fig. 1 A free hanging submersed cable segment in static equilibrium

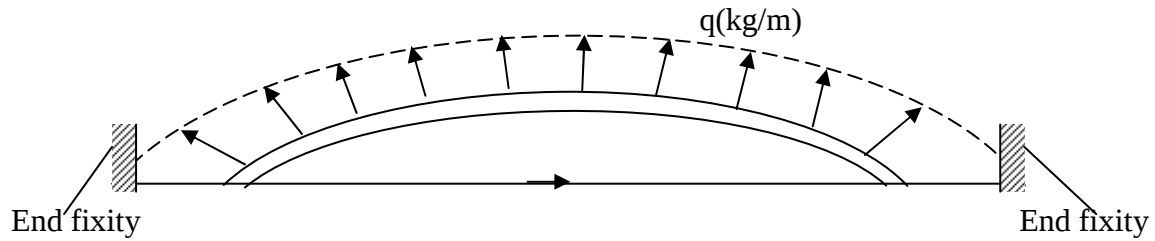


Fig. 2 Generic representation of a submersed cable

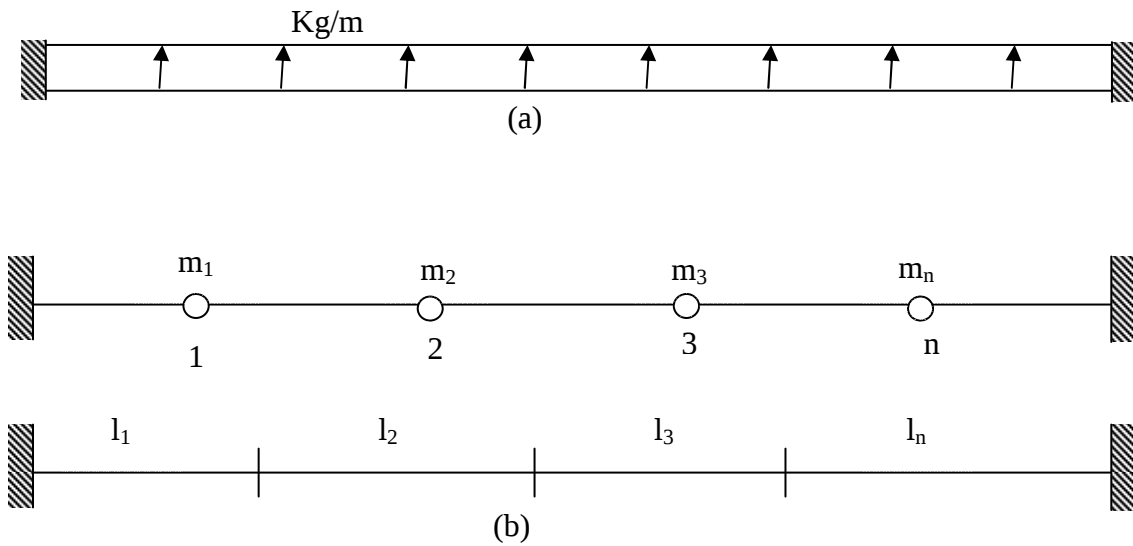


Fig. 3: A submersed cable with distributed mass and associated lumped mass model

Consider a submersed cable with distributed mass lumped at points 1, 2, ..., n respectively (Figure 3). At an arbitrary time t , the displacement of the masses m_i ($i = 1, 2, \dots, n$) at the chosen nodal points is given by

$$y_i(t) + A_i \sin \omega t, i = 1, 2, \dots, n \quad (1)$$

The motion of the submersed cable is said to be defined if the expressions for $X_1, X_2, \dots, X_n$ are known at an arbitrary time in the course of the system's motion. The forces acting on the cable are

- i) Inertia force due to mass m_1
- ii) Inertia force due to mass m_2
- iii) Inertia force due to mass m_n

For self-excited vibration of a submersed cable with distributed mass substituted by limped masses $m_1, m_2, \dots, m_n$ and constant tension P_o , the equations of motion are;

$$\left. \begin{aligned} -y_1 &= m_1 \ddot{y}_1 \delta_{11} + m_2 \ddot{y}_2 \delta_{12} + \dots m_n \ddot{y}_n \delta_{1n} \\ -y_2 &= m_1 \ddot{y}_1 \delta_{21} + m_2 \ddot{y}_2 \delta_{22} + \dots m_n \ddot{y}_n \delta_{2n} \\ -y_n &= m_1 \ddot{y}_1 \delta_{n1} + m_2 \ddot{y}_2 \delta_{n2} + \dots m_n \ddot{y}_n \delta_{nn} \end{aligned} \right\} \quad (2)$$

In matrix form, equation (2) is represented as

$$-\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \partial_{11} & \partial_{12} & \dots & \partial_{1n} \\ \partial_{21} & \partial_{22} & \dots & \partial_{2n} \\ \dots & \dots & \dots & \dots \\ \partial_{n1} & \partial_{n2} & \dots & \partial_{nn} \end{bmatrix} \cdot \begin{bmatrix} m_1 \ddot{y}_1 \\ m_2 \ddot{y}_2 \\ \vdots \\ m_n \ddot{y}_n \end{bmatrix} \quad (3)$$

where $\begin{bmatrix} m \ddot{y} \end{bmatrix}$ = Inertia force matrix
 $[\delta_{ij}], [\delta_{ii}]$ = flexibility influence coefficient matrices
 δ_{ij} = Deflection at point i due to a unit load applied at point j
 δ_{ii} = Deflection at point i due to a unit load applied at point i

From Maxwell reciprocal theorem, $\delta_{ij} = \delta_{ji}$

The displacement at chosen nodal points and their second derivatives are now substituted into obtained equation of motion (equation 3) to give the cable frequency equations as follows.

$$\begin{aligned} y_i^1(t) &= A\omega \cos \omega t \\ y_i''(t) &= A\omega^2 \sin \omega t \end{aligned} \quad (4)$$

$$-\begin{bmatrix} A_1 \sin \omega t \\ A_2 \sin \omega t \\ \vdots \\ A_n \sin \omega t \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} \\ \dots & \dots & \dots & \dots \\ \delta_{b1} & \delta_{b2} & \dots & \delta_{NN} \end{bmatrix} \begin{bmatrix} -m_1 \omega^2 A_1 \sin \omega t \\ -m_2 \omega^2 A_2 \sin \omega t \\ \vdots \\ -m_n A_n \sin \omega t \end{bmatrix} \quad (5)$$

For two-degree of freedom submerged cable system, the frequency equation (equation 5) reduces to

$$\begin{aligned} -A_1 \sin \omega t &= -m_1 A_1 \omega \sin \omega t \delta_{11} - m_2 A_2 \omega^2 \sin \omega t \delta_{12} \\ -A_2 \sin \omega t &= -m_1 A_1 \omega^2 \sin \omega t \delta_{21} - m_2 A_2 \omega^2 \sin \omega t \delta_{22} \end{aligned} \quad (6)$$

Multiplying both sides of equation (6) by -1 gives:

$$\begin{aligned} A_1 \sin \omega t &= m_1 A_1 \omega \sin \omega t \delta_{11} + m_2 A_2 \omega^2 \sin \omega t \delta_{12} \\ A_2 \sin \omega t &= -m_1 A_1 \omega^2 \sin \omega t \delta_{21} + m_2 A_2 \omega^2 \sin \omega t \delta_{22} \end{aligned} \quad (7)$$

$$\begin{aligned} m_1 A_1 \omega^2 \sin \omega t \delta_{11} - A_1 \sin \omega t + m_2 A_2 \omega^2 \sin \omega t \delta_{12} \\ m_1 A_1 \omega^2 \sin \omega t \delta_{21} + m_2 A_2 \omega^2 \sin \omega t \delta_{22} - A_2 \sin \omega t &= 0 \end{aligned} \quad (8)$$

or

$$\left. \begin{aligned} (m_1 \omega^2 \delta_{11} - 1) A_1 + m_2 A_2 \omega^2 \delta_{12} &= 0 \\ m_1 \omega^2 \delta_{21} A_1 + (m_2 \omega^2 \delta_{22} - 1) A_2 &= 0 \end{aligned} \right\} \quad (9)$$

Equation (9) represents the frequency equation for a submerged cable subjected to two degree of freedom under self excited vibration.

Dividing equation (9) above by $m_i \omega^2$ transforms equation (9) to

$$\left. \begin{aligned} \left(\delta_{11} - \frac{1}{m_i \omega^2} \right) A_1 + \delta_{12} A_2 &= 0 \\ \delta_{21} A_1 + \left(\delta_{22} - \frac{1}{m_i \omega^2} \right) A_2 &= 0 \end{aligned} \right\} \quad (10)$$

Putting equation (10) in matrix form gives

$$\begin{bmatrix} \delta_{11} - \frac{1}{m_i \omega^2} & \delta_{12} \\ \delta_{21} & \delta_{22} - \frac{1}{m_i \omega^2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11)$$

If $m_1 = m_2 = m$

and

$$\frac{1}{m\omega^2} = \lambda \quad (12)$$

The frequency equation (equation 10) now transforms to

$$\begin{bmatrix} \delta_{11} - \lambda & \delta_{12} \\ \delta_{21} & \delta_{22} - \lambda \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13)$$

For non-trivial solution, the determinant of the frequency equation (equation 13) is zero

$$\text{Det} \begin{vmatrix} \delta_{11} - \lambda & \delta_{12} \\ \delta_{21} & \delta_{22} - \lambda \end{vmatrix} = 0 \quad (14)$$

Simplifying equation (14) gives:

$$(\delta_{11} - \lambda)(\delta_{22} - \lambda) - \delta_{21}\delta_{12} = 0 \quad (15)$$

The roots of equation (15) gives the natural frequencies of a submersed cable under self-excited vibration.

RESULTS AND DISCUSSION

Numerical Example

A submersed cable of mass density ρ has a length of 15m. Determine the first two natural frequencies of the cable under self-excited vibration.

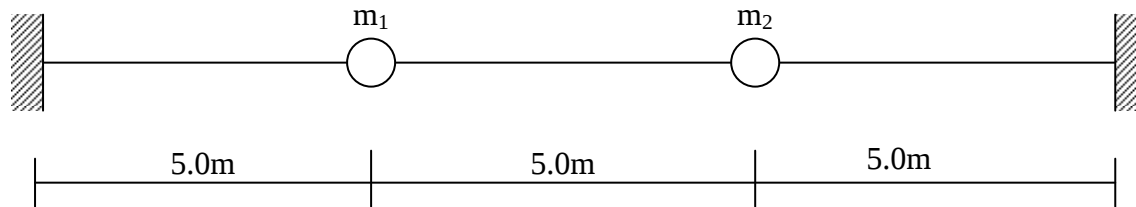
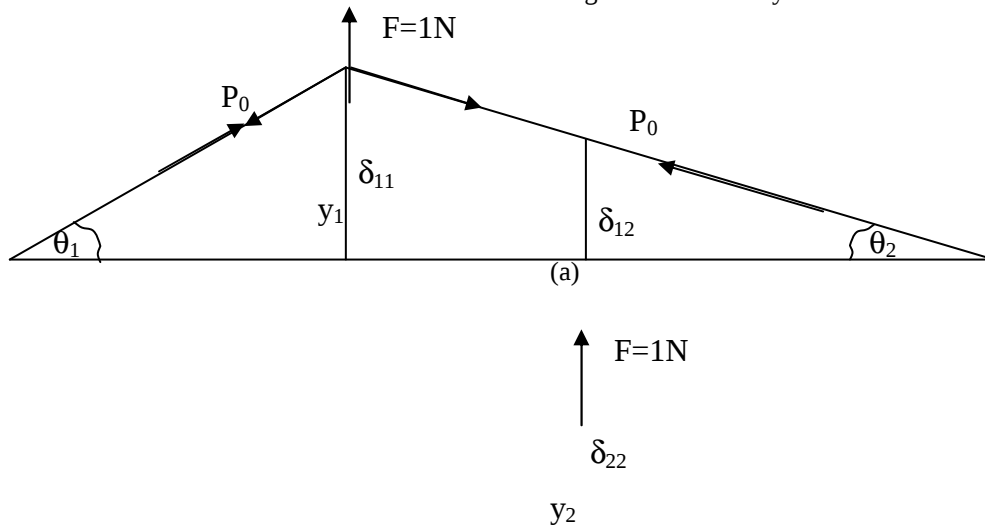


Fig. 4 A submersed cable modeled as a two-degree of freedom system



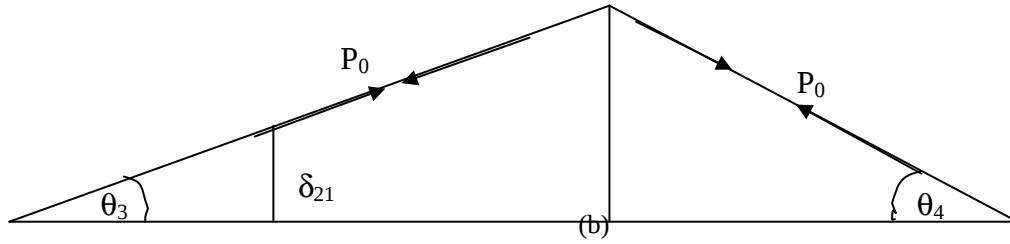


Fig. 5 Derivation of influence coefficients

From figure 1(a),

$$F = P_O(\sin\theta_1 + \sin\theta_2) \quad (16)$$

For small angle $\sin\theta_1 = \theta_1$ and $\sin\theta_2 = \theta_2$

$$F = P_O(\theta_1 + \theta_2) \quad (17)$$

In terms of displacement of lumped masses at nodal point 1,

$$F = P_o \left(\frac{y_1}{5} + \frac{y_1}{10} \right) = P_o \left(\frac{3y_1}{10} \right)$$

$$1 = P_o \left(\frac{3P_o}{10} \right)$$

$$\frac{1}{y_1} = \frac{3P_o}{10}$$

$$y_1 = \delta_{11} = \frac{10}{3P_o}$$

$$\delta_{12} = \delta_{21} = \delta_{11/2} = \frac{10}{6P_o} = \frac{5}{3P_o}$$

From Fig. 1(b),

$$F = P_o(\sin\theta_3 + \sin\theta_4) \quad (18)$$

Again, for small angle,

$$\sin\theta_3 = \theta_3 \text{ and } \sin\theta_4 = \theta_4$$

$$F = P_o(\theta_3 + \theta_4) \quad (19)$$

At nodal point 2,

$$F = P_o \left(\frac{y_2}{10} + \frac{y_2}{5} \right) = P_o \left(\frac{3y_2}{10} \right)$$

$$1 = P_o \left(\frac{3y_2}{10} \right)$$

$$\frac{1}{y_2} = \frac{3P_o}{10} \Rightarrow y_2 = \delta_{22} = \frac{10}{3P_o}$$

Putting the obtained influence coefficients in the equation (15) gives

$$\left(\frac{10}{3P_o} - \lambda \right) \left(\frac{10}{3P_o} - \lambda \right) - \frac{5}{3P_o} \bullet \frac{5}{3P_o} = 0 \quad (20)$$

$$\Rightarrow \left(\frac{10}{3P_o} - \lambda \right)^2 - \frac{25}{9P_o^2} = 0$$

$$\Rightarrow \left(\frac{10}{3P_o} - \lambda \right)^2 = \frac{25}{9P_o^2}$$

$$\frac{10}{3P_o} - \lambda = \pm \frac{5}{3P_o}$$

$$\lambda_1 = \frac{10}{3P_o} + \frac{5}{3P_o} = \frac{15}{3P_o}$$

$$\lambda_2 = \frac{10}{3P_o} - \frac{5}{3P_o} = \frac{5}{3P_o}$$

$$\text{But } \lambda_i = \frac{1}{m\omega^2}$$

$$\Rightarrow \frac{1}{m\omega_1^2} = \frac{15}{3P_o}$$

$$\Rightarrow \omega_1 = \sqrt{\frac{3P_0}{15m}} = 0.4472 \sqrt{\frac{P_0}{m}} \text{ rad / s}, \quad \omega_2 = \sqrt{\frac{3P_0}{5m}} = 0.7746 \sqrt{\frac{P_0}{m}} \text{ rad / s}$$

The exact solution (James, 1984) is

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{P_0}{\rho}} \quad (21)$$

$$n = 1, 2, \dots, n$$

where ρ = mass density of the submerged cable.

From statical consideration,

$$m_1 = m_2 = m = \frac{\rho(5+5)}{2} = \frac{10\rho}{2} = 5\rho$$

$$\Rightarrow m_1 = m_2 = 5\rho \quad (22)$$

Equation (21) now becomes

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{5P_0}{m}} \quad (23)$$

$$\omega_1 = \frac{1 \times \pi}{15} \sqrt{\frac{5P_0}{m}} = 0.4684 \sqrt{\frac{P_0}{m}} \text{ rad / sec}$$

$$\omega_2 = \frac{2 \times \pi}{15} \sqrt{\frac{5P_0}{m}} = 0.9368 \sqrt{\frac{P_0}{m}} \text{ rad / sec}$$

Table 1: Comparison of results of the present model with exact values

Natural frequency	$\omega_1(\text{rad/s})$	$\omega_2(\text{rad/s})$
Present model	$0.4472 \sqrt{\frac{P_0}{m}}$	$0.7746 \sqrt{\frac{P_0}{m}}$
Exact	$0.4684 \sqrt{\frac{P_0}{m}}$	$0.9368 \sqrt{\frac{P_0}{m}}$

DISCUSSION OF RESULT AND CONCLUSION

From Table 1, it can be seen that the present model which is based on lumped mass idealization produced results that agreed favourably with the simulated values. The difference in the response values may be due to difficulty inherent in the derivation of the present model.

In conclusion, the present model can be used to simulate the actual system (James, 1984). The mathematical process involved is simple and less rigorous most especially when a finite number of lumped masses is involved.

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